Nonparametric predictive inference for voting systems

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Abstract

We present upper and lower probabilities for reliability of voting systems, also known as k-out-of-m systems, which include series- and parallel-systems. We restrict attention to systems with identical components. These interval probabilities are based on the nonparametric predictive inferential (NPI) approach for Bernoulli data presented by Coolen (1998). In this approach, it is assumed that test data are available on the components, and that the future components to be used in the system are exchangeable with these.

This approach fits into the general framework of Imprecise Reliability, which has received increasing attention in recent years (Utkin and Coolen 2007). An early overview of NPI in Reliability was presented by Coolen, Coolen-Schrijner and Yan (2002), in recent years NPI has been further developed and presented as an attractive statistical theory for situations where one aims at inference for future observables on the basis of available data, adding only rather limited additional assumptions (Coolen 2006a). A particularly attractive feature of NPI in Reliability, with lower and upper probabilities, is that data containing zero failures can be dealt with in an attractive manner, which will also be illustrated in this paper for reliability of voting systems.

1 Introduction

During the last decade, imprecise probabilistic methods in reliability have received increasing attention, concise overviews are presented by Coolen (2004) and by Coolen and Utkin (2007) while a detailed introduction and overview is given by Utkin and Coolen (2007). These methods are based on generalized uncertainty quantifications via lower and upper probabilities, also known as theory of imprecise probability (Walley 1991) or interval probability (Weichselberger 2000, Weichselberger 2001). During this period, Coolen and several collaborators (for a brief overview, see Coolen 2006a) have developed a novel statistical theory entitled Nonparametric Predictive Inference (NPI), with an early overview of possible applications of NPI in reliability presented by Coolen, Coolen-Schrijner and Yan (2002). Recently presented applications of NPI to reliability problems include reliability demonstration for failure-free periods (Coolen and Coolen-Schrijner 2005), comparison of success-failure data (Coolen-Schrijner and Coolen 2007) and probabilistic safety assessment in case of zero failures (Coolen 2006b).

A traditional problem considered in reliability theory is assessment of systems reliability (Andrews and Moss 2002), where voting systems have received particular attention. These are systems that consist of m exchangeable components (often the confusing term 'identical components' is used), such that the system functions if and only if at least k of its components function, with series systems ($k = m$) and parallel systems ($k = 1$) as special cases. Such systems are also known as 'k-out-of-m systems'. Utkin (2004) considered such systems for situations with incomplete information, using imprecise probability to quantify uncertainty. In this paper, we apply Coolen's NPI method for Bernoulli data to voting systems, where we assume that inferences on system reliability are based on information from tests on n components, which are exchangeable with the components in the system considered. Throughout, we only consider situations where components, and therefore also the system, either function or not when called upon, so we are not studying failure behaviour over time. Generally, we will assume that a test of

n components revealed that s of them functioned and $n-s$ failed, and we will use NPI to derive the lower and upper probabilities for the event that the k-out-of-m system, made up of components exchangeable with those n tested, functions.

In Section 2 we give a brief overview of NPI, and particularly of NPI for Bernoulli random quantities as used later in this paper. Section 3 presents the main results on NPI for voting systems, and these results are illustrated and discussed via examples in Section 4. Section 5 provides some concluding remarks, including a brief outline of related research challenges.

2 Nonparametric Predictive Inference

In this section, we summarize results from Coolen (1998) on NPI for Bernoulli random quantities, we refer to Coolen (1998) for justifications, which are based on a representation of Bernoulli data as outcomes of an experiment similar to that used by Bayes (1763), with Hill's assumption $A_{(n)}$ (Hill 1968, Hill 1988) used to derive direct predictive probabilities (Dempster 1963, Geisser 1993) for future observations using available data. The lower and upper probabilities presented by Coolen (1998) fit in the framework of nonparametric predictive inference (NPI) (Augustin and Coolen 2004, Coolen 2006a), hence we also call them NPI (-based) lower and upper probabilities, and they have strong internal consistency properties in the theory of interval probability (Augustin and Coolen 2004, Weichselberger 2001) as can be proven similarly to the proofs of such consistency properties for the more general situation of multinomial data (Coolen and Augustin 2005). Due to the use of $A_{(n)}$ in deriving these lower and upper probabilities, they fit in a frequentist framework of statistics but can also be interpreted from Bayesian perspective (Hill 1988, Hill 1993). As they are conditional lower and upper probabilities which are introduced without reference to probabilities for the unconditional events, they can be interpreted in a way similar to Dempster's 'direct probabilities' (Dempster 1963). NPI is also 'perfectly calibrated' in the sense of Lawless and Fredette (2005). For further discussion of such inferences see Augustin and Coolen (2004) and Coolen (2006a).

Suppose that we have a sequence of $n+m$ exchangeable Bernoulli trials, each with 'success' and 'failure' as possible outcomes, and data consisting of s successes in n trials. Let Y_1^n denote the random number of successes in trials 1 to n, then a sufficient representation of the data for our inferences is $Y_1^n = s$, due to the assumed exchangeability of all trials. Let Y_{n+1}^{n+m} denote the random number of successes in trials $n+1$ to $n+m$. Let $R_t = \{r_1, \ldots, r_t\}$, with $1 \leq t \leq m+1$ and $0 \leq r_1 < r_2 < \ldots < r_t \leq m$, and, for ease of notation, let us define $\binom{s+r_0}{s} = 0$. Then the NPI-based upper probability (Coolen 1998) for the event $Y_{n+1}^{n+m} \in R_t$, given data $Y_1^n = s$, for $s \in \{0, \ldots, n\}$, is

$$\overline{P}(Y_{n+1}^{n+m} \in R_t | Y_1^n = s) = \binom{n+m}{n}^{-1} \sum_{j=1}^{t} \left[\binom{s+r_j}{s} - \binom{s+r_{j-1}}{s} \right] \binom{n-s+m-r_j}{n-s}$$

The corresponding lower probability (Coolen 1998) is derived via the conjugacy property

$$\underline{P}(Y_{n+1}^{n+m} \in R_t | Y_1^n = s) = 1 - \overline{P}(Y_{n+1}^{n+m} \in R_t^c | Y_1^n = s)$$

where $R_t^c = \{0, 1, \ldots, m\} \backslash R_t$. This conjugacy property between these upper and lower probabilities is justified in Coolen (1998), and agrees with the fact that these are F-probabilities in Weichselberger's theory of interval probability (Weichselberger 2001). These lower and upper probabilities also have attractive properties beyond internal consistency, as the interval created by the lower and upper probability for an event A always contains the precise empirical probability for A as based on the observed data, and the lower (upper) probability increases (decreases) as function of n, for constant s/n.

3 NPI for voting systems

In this paper, we focus on the lower and upper probabilities for the event $Y_{n+1}^{n+m} \geq k$, as this event corresponds to successful functioning of a k-out-of-m system. Given data consisting of s successes from n

components tested, we also denote these lower and upper probabilities for the event that the k-out-of-m system functions successfully, by $\underline{P}(m:k|n,s)$ and $\overline{P}(m:k|n,s)$, respectively. From the upper probability for $Y_{n+1}^{n+m} \in R_t$ given in Section 2 (Coolen 1998), we derive, for $k \in \{1,2,\ldots,m\}$ and $0 < s < n$, the upper probability

$$\begin{aligned}\overline{P}(m:k|n,s) &= \overline{P}(Y_{n+1}^{n+m} \geq k | Y_1^n = s) \\ &= \binom{n+m}{n}^{-1} \left[\binom{s+k}{s} \binom{n-s+m-k}{n-s} + \sum_{l=k+1}^{m} \binom{s+l-1}{s-1} \binom{n-s+m-l}{n-s} \right] \quad (1)\end{aligned}$$

and, via the conjugacy property, the lower probability

$$\begin{aligned}\underline{P}(m:k|n,s) &= \underline{P}(Y_{n+1}^{n+m} \geq k | Y_1^n = s) \\ &= 1 - \binom{n+m}{n}^{-1} \left[\sum_{l=0}^{k-1} \binom{s+l-1}{s-1} \binom{n-s+m-l}{n-s} \right] \quad (2)\end{aligned}$$

For $m = 1$, so considering a system consisting of a single component, the upper and lower probabilities for the event that the system functions successfully are $\overline{P}(1:1|n,s) = \overline{P}(Y_{n+1}^{n+1} = 1 | Y_1^n = s) = \frac{s+1}{n+1}$ and $\underline{P}(1:1|n,s) = \underline{P}(Y_{n+1}^{n+1} = 1 | Y_1^n = s) = \frac{s}{n+1}$.

If the observed data are all successes, so $s = n$, or all failures, so $s = 0$, then the upper probabilities are, for all $k \in \{1,\ldots,m\}$,

$$\begin{aligned}\overline{P}(m:k|n,n) &= \overline{P}(Y_{n+1}^{n+m} \geq k | Y_1^n = n) = 1 \\ \overline{P}(m:k|n,0) &= \overline{P}(Y_{n+1}^{n+m} \geq k | Y_1^n = 0) = \binom{n+m-k}{n} \binom{n+m}{n}^{-1}\end{aligned}$$

and the lower probabilities are, for all $k \in \{1,\ldots,m\}$,

$$\begin{aligned}\underline{P}(m:k|n,n) &= \underline{P}(Y_{n+1}^{n+m} \geq k | Y_1^n = n) = 1 - \binom{n+k-1}{n} \binom{n+m}{n}^{-1} \\ \underline{P}(m:k|n,0) &= \underline{P}(Y_{n+1}^{n+m} \geq k | Y_1^n = 0) = 0\end{aligned}$$

For the two extreme cases of series and parallel systems, with $k = m$ and $k = 1$, respectively, these upper and lower probabilities can be substantially simplified to give the expressions below, which actually provide much insight into the NPI approach for such systems. Representing corresponding lower and upper probabilities for an event A by $(\underline{P}, \overline{P})(A)$, the general results above are, for series systems,

$$\begin{aligned}(\underline{P}, \overline{P})(m:m|n,0) &= \left(0, \prod_{j=1}^{m} \frac{j}{n+j}\right) \\ (\underline{P}, \overline{P})(m:m|n,s) &= \left(\prod_{j=1}^{m} \frac{s-1+j}{n+j}, \prod_{j=1}^{m} \frac{s+j}{n+j}\right) \quad \text{for } 0 < s < n \\ (\underline{P}, \overline{P})(m:m|n,n) &= \left(\frac{n}{n+m}, 1\right)\end{aligned}$$

and for parallel systems,

$$
\begin{aligned}
(\underline{P},\overline{P})(m:1|n,0) &= \left(0,\frac{m}{n+m}\right) \\
(\underline{P},\overline{P})(m:1|n,s) &= \left(1-\prod_{j=1}^{m}\frac{n-s+j}{n+j},1-\prod_{j=1}^{m}\frac{n-s-1+j}{n+j}\right) \quad \text{for } 0<s<n \\
(\underline{P},\overline{P})(m:1|n,n) &= \left(1-\prod_{j=1}^{m}\frac{j}{n+j},1\right)
\end{aligned}
$$

In these expressions, we have e.g. written $\binom{n+m}{n}^{-1}$ as $\prod_{j=1}^{m}\frac{j}{n+j}$, as the latter expression provides more insight. For example, it is the value of the upper probability of successful functioning of a series system when all the n components tested failed, $\overline{P}(m:m|n,0)$. In this situation, when we consider the m components of the system in sequence, then the upper probability of the first to function, given the data, is $\frac{1}{n+1}$. If this first component in the system were to function, then we would use the data of the n tested components together with this functioning first component to infer the upper probability that the second component in the system functions, given that the first functions, to be equal to $\frac{2}{n+2}$, and so on for all m components in sequence, with for each one both the information from the test and on the previous components of the system in this sequence used. It is easily seen that this leads to $\overline{P}(m:m|n,0)=\prod_{j=1}^{m}\frac{j}{n+j}$. The other expressions for these series and parallel systems can be derived and interpreted via similar direct considerations.

An important advantage of the use of lower and upper probabilities in statistical inference occurs in situations with the observations either all successes or all failures, as inferences based on precise probabilistic methods are typically not in agreement with empirical probabilities in such cases. For example, if one has observed zero failures in tests of n components, one might expect a future component to have a small probability of failure, and one may not wish to assign 0 to this probability for any finite value of n. However, zero failures in n components does not exclude the possibility that failures could never happen (e.g. one might have defined 'failure' in a manner that is not relevant for the functioning of the component). An attractive, albeit informal, manner in which to interpret lower and upper probabilities is to regard the lower probability $\underline{P}(A)$ as quantifying the evidence in favour of event A, and the upper probability $\overline{P}(A)$ as quantifying the evidence against event A (hence in favour of A^c, in agreement with the conjugacy property). From this perspective, if one considers a system consisting of only a single component, the lower and upper probabilities of successful functioning of the one future component, given zero failures in n components tested (so $s=n$), which are equal to $(\underline{P},\overline{P})(1:1|n,n)=(\frac{n}{n+1},1)$, are attractive, as the upper probability reflects that there is no evidence from the test data against successful functioning of the future component, whereas the lower probability provides a natural cautious inference which can be used in quantitative risk assessment. As such, the results in this paper can be used in zero-failure reliability demonstration from NPI perspective, generalizing the results presented by Coolen and Coolen-Schrijner (2005). For example, one can consider decisions on levels of redundancy to build into the system (e.g. the value of m if k is determined by the system requirements), if possible, in order to reduce the number of zero-failure component tests required to demonstration reliability at a chosen level. This is briefly illustrated in Example 3 in Section 4. If one also wishes to take costs of components and of testing into account, together with practical constraints on test budget and time, then the results in this paper can be used in a straightforward manner, in line with the Bayesian approach to such problems (Rahrouh, Coolen and Coolen-Schrijner 2006).

For the special cases with $m=1$, $k=1$ or $k=m$, for which the lower and upper probabilities of successful system functioning given s successes in n component tests are given above, it is easily seen

that the following result holds, for $0 \leq s < n$

$$\overline{P}(m : k|n, s) = \underline{P}(m : k|n, s+1) \tag{3}$$

The result (3) actually holds generally for all k-out-of-m systems as considered in this paper. A direct proof, using the expressions (1) and (2), is a complicated exercise in combinatorial analysis. However, this result follows immediately from detailed consideration of the underlying representation assumed for Bernoulli random quantities in the NPI method by Coolen (1998) that is used here. In this paper, we do not provide this detailed justification for the equality (3), but the examples in Section 4 will, of course, illustrate this interesting property of our inferences. A second way to quickly justify the equality (3) is based on the fact that all NPI results for k-out-of-m systems presented in this section can also be derived within the Bayesian framework, which will be briefly discussed in Section 5. The result (3) can obviously be used to reduce computational effort, if upper and lower probabilities are required for all possible values of s. We would also like to emphasize the elegance of this equality, as it implies that the intervals created by corresponding lower and upper probabilities of successful system functioning, for $s = 0, 1, \ldots, n$, form a partition of the interval $[0, 1]$.

4 Examples

In this section we provide numerical examples to illustrate the results presented in Section 3, and we discuss some related issues.

Example 1.
Consider a series system with 10 exchangeable components (so $k = m = 10$), and the only information available is the result of a test of 2 components, also exchangeable with the 10 to be used in the system. For the three possible values of the number of successes in the tests, $s = 0, 1, 2$, the NPI lower and upper probabilities for successful functioning of the system are

$$\begin{aligned}
(\underline{P}, \overline{P})(10 : 10|2, 0) &= \left(0, \frac{1}{66}\right) \\
(\underline{P}, \overline{P})(10 : 10|2, 1) &= \left(\frac{1}{66}, \frac{1}{6}\right) \\
(\underline{P}, \overline{P})(10 : 10|2, 2) &= \left(\frac{1}{6}, 1\right)
\end{aligned}$$

These values illustrate some of the general properties presented in Section 3, namely that the upper probability of successful system functioning given s successes in n tests is equal to the lower probability of successful system functioning given $s + 1$ successes in n tests. The value 0 (1) of the lower (upper) probability for the case $s = 0$ ($s = 2$) reflects that in this case there is no strong evidence that the components can actually function (fail). These values emphasize the serious error that can be made if, instead of a careful analysis which combines the uncertainty modelling and available information, one would plug a 'reasonable' estimate of a parameter representing functioning of a component into a formula for a probability of system functioning depending on this parameter. For example, if one would assume a parameter θ representing an unknown probability of such a component to function, then conditional on this parameter, the probability of successful functioning of this series system would be θ^{10}. For $s = 1$, so one failure and one success in two tested components, a 'reasonable' estimate would be $\hat{\theta} = 1/2$ (e.g. this would be both the moment estimate and maximum likelihood estimate), and one could be tempted to use this value to predict successful functioning of the series with probability $\hat{\theta}^{10} = (1/2)^{10} = 1/1024$, which is far lower than the corresponding lower probability $\frac{1}{66}$ from our method, so system reliability would be substantially under-estimated. Although it is well known that plugging point estimates into such formulae is wrong, text books in reliability rarely address such issues carefully. In Section 5 we will briefly

comment on the Bayesian approach, where it is easily seen that, for any reasonably non-informative prior distribution, the corresponding predictive probability of system functioning is within the interval created by our lower and upper probabilities. As an informal argument that leads to a better alternative than the plug-in approach, and that is in line with our approach, we could reason as follows. On the basis of one success in two tests, the predictive probability for the next component to be successful might be set at (about) 1/2. For the series system to function successfully, all 10 components must function. Let us then consider the second component in the series system, conditional on the first functioning successfully and the data from the test, hence for this component the information available consists of 2 successful components out of 3, and therefore the predictive probability for this component to function might be set at (about) 2/3. Continuing this reasoning, which acknowledges the interdependence of the 10 components in the system, the predictive probability of successful functioning of the series system would be (about)

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \ldots \times \frac{10}{11} = \frac{1}{11}$$

which is in between our corresponding lower and upper probabilities. We should emphasize that we only present this latter informal reasoning as a possible explanation on why the use of a plug-in estimate is wrong, we do not suggest that the value 1/11 is a 'correct' precise probability in this case, an obvious reason for this is that this informal argument would lead to precise probability 0 (1) for system functioning in the case $s = 0$ ($s = 2$).

If, instead of a series system, we consider a parallel system with 10 components (so $k = 1$, $m = 10$) with 2 components tested, then the NPI lower and upper probabilities of successful functioning of the system are

$$\begin{aligned}
(\underline{P}, \overline{P})(10:1|2,0) &= \left(0, \frac{5}{6}\right) \\
(\underline{P}, \overline{P})(10:1|2,1) &= \left(\frac{5}{6}, \frac{65}{66}\right) \\
(\underline{P}, \overline{P})(10:1|2,2) &= \left(\frac{65}{66}, 1\right)
\end{aligned}$$

Note here that, if one mistakenly used a plug-in estimate of 1/2 for the case $s = 1$, as discussed above, one could think that the value $1 - (1/2)^{10} = 1023/1024$ would be a reasonable estimate of the system reliability. This value is substantially higher than the NPI upper probability 65/66, and far greater than the NPI lower probability 5/6 which one might wish to use for risk assessment from a cautious perspective, and hence there could be a danger of over confidence in the system's reliability if the uncertainty and information are not properly analysed.

Example 2.
To further illustrate the NPI results for system reliability, presented in Section 3, Table 1 provides all lower and upper probabilities for the possible cases with $n = 4$ components tested, of which s functioned successfully, and the system consisting of $m = 5$ components, of which at least k must function.

The values in Table 1 show that, in order to get a reasonably large lower probability of successful system functioning, one does not necessarily require most tested components to have functioned well if k is small, which means that the system has much build-in redundancy, but for large values of k one requires (nearly) all tested components to have been successful. Example 3 briefly discusses the related issue of possible choice between extra testing or extra system redundancy for reliability demonstration.

Table 1: Lower and upper probabilities for all cases with $m = 5$ and $n = 4$

	$k=1$		$k=2$		$k=3$		$k=4$		$k=5$	
	$\underline{P}$	$\overline{P}$	$\underline{P}$	$\overline{P}$	$\underline{P}$	$\overline{P}$	$\underline{P}$	$\overline{P}$	$\underline{P}$	$\overline{P}$
$s=0$	0	0.556	0	0.278	0	0.119	0	0.040	0	0.008
$s=1$	0.556	0.833	0.278	0.595	0.119	0.357	0.040	0.167	0.008	0.048
$s=2$	0.833	0.952	0.595	0.833	0.357	0.643	0.167	0.405	0.048	0.167
$s=3$	0.952	0.992	0.833	0.960	0.643	0.881	0.405	0.722	0.167	0.444
$s=4$	0.992	1	0.960	1	0.881	1	0.722	1	0.444	1

Example 3.
As mentioned in Section 3, the results in this paper can also be used in zero-failure reliability demonstration from NPI perspective, generalizing the results presented by Coolen and Coolen-Schrijner (2005). Suppose that for system functioning it is required that k components function, but that redundancy can be built into the system by increasing the total number of components m in the system. For example, components considered could be batteries required to provide back up in case of problems with electricity supply for a safety-critical system, where system functioning requires a minimum of three batteries to function when demanded, but where installing more batteries might provide important redundancy. Rahrouh, Coolen and Coolen-Schrijner (2006) presented a Bayesian approach for optimal decisions for reliability demonstration, assuming that only component tests with zero failures would lead to release of the system for practical use, as is often the case if high reliability is required. They considered both costs of testing and costs of extra system redundancy, and also took practical constraints with regard to test time and budget into account. Apart from cost and time figures, and related constraints, the key input for such decisions consists of the lower probabilities $\underline{P}(m : k|n, n)$, as function of m and n for fixed k. Some such values are presented in Table 2, for $k = 8$ and the cases $n = 5, 10, 15$, and m varying from 8 to 12. Of course, the corresponding upper probabilities are all equal to one, as the tests revealed zero failures.

Table 2: Lower probabilities for zero-failure testing with $k = 8$

	$m=8$	9	10	11	12
$s=n=5$	0.385	0.604	0.736	0.819	0.872
10	0.556	0.789	0.895	0.945	0.970
15	0.652	0.870	0.948	0.978	0.990

The lower probabilities presented in Table 2 can be used in several ways. For example, consider the case $m = 8$ with 5 zero-failure tests, leading to lower probability 0.385 of successful system functioning. The table shows that increasing the redundancy to $m = 9$, keeping $k = 8$, would increase the lower probability to 0.604, while increasing the number of zero-failure tests to 10 would increase the lower probability to 0.556, so if these two actions were available at similar costs, increase of redundancy might be preferred to more tests. However, if 15 tests were possible at a cost similar to the added redundancy, then this might be preferred as the corresponding lower probability would increase to 0.652, if all 15 tests would be successes. Of course, extra testing has the added advantage of possibly finding more failures, in which case one would start the analysis over again after further inspection or development of the components. In our NPI approach, the absence of prior information makes it impossible to infer how likely failures in the tests would be, but in high reliability demonstration one would normally be quite surprised to encounter failures in tests.

Table 3 extends this example by presenting the minimum number of zero-failure tests required to

achieve a chosen value for the lower probability of successful system functioning, again for $k = 8$ and m varying from 8 to 12. The requirement considered is $\underline{P}(m : 8|n, n) \geq p$ for different values of p.

Table 3: Required number n of zero-failure tests for $\underline{P}(m : 8|n, n) \geq p$

	$m = 8$	9	10	11	12
$p = 0.75$	24	9	6	4	4
0.80	32	11	7	5	4
0.85	46	14	8	6	5
0.90	72	19	11	8	6
0.95	153	30	16	11	9
0.99	792	77	33	21	15

The main conclusion to draw from Table 3 is that, in order to demonstrate high reliability via zero-failure testing, one requires quite a large number of successful tests, yet this number can be substantially reduced by building in redundancy.

5 Concluding remarks

The Bayesian approach to statistics also provides a natural framework for inferences of the kind considered in this paper. If one assumes a parameter θ representing the probability of successful functioning of a single component, then the probability of successful functioning of a k-out-of-m system, as a function of θ, is simply represented by

$$P(m : k|\theta) = \sum_{j=k}^{m} \binom{m}{k} \theta^k (1-\theta)^{m-k}$$

For any assumed prior probability distribution for θ and test data, the Bayesian approach leads to a precise posterior probability for successful functioning of the system. Beta distributions are particularly attractive prior distributions for θ in this case, as they are conjugate, which means that the corresponding posterior distributions θ are also Beta distributions. The results presented in this paper actually coincide with the corresponding Bayesian results based on two particular Beta prior distributions, namely the NPI lower probabilities of successful system functioning correspond to Bayesian probabilities based on the Beta$(0, 1)$ prior, and the NPI upper probabilities correspond similarly to the Beta$(1, 0)$ prior (note that these priors are improper, but the corresponding posterior probabilities of interest do exist). This is due to the fact that, generally, for events of the form 'k or more successes out of m trials', the inferences of Coolen (1998) coincide with these Bayesian inferences. It should, however, be emphasized that this is not the case for all events considered in the NPI approach by Coolen (1998). The fact that these inferences provide the same values for the (lower and upper) probabilities considered can be understood from the representation of successes and failures that underlies NPI (Coolen 1998), and which is closely related to the approach by Bayes (1763). This underlying representation, which we do not discuss further here, also provides a simple justification for the equality (3) presented in Section 3. In relation to Example 1, it is useful to remark that, if one would use a Bayesian approach with improper prior Beta$(0, 0)$, and add test data consisting of one success and one failure (leading to a uniform posterior distribution for θ), then the posterior probability of successful functioning of a 10-out-of-10 system would be equal to $1/11$, the value also derived via an informal argument in Example 1.

Hartigan (1983) proposed the use of either the Beta$(0, 1)$ or the Beta$(1, 0)$ prior for 'cautious' inference, and we also proposed the Beta$(0, 1)$ prior for Bayesian high-reliability demonstration, mainly due to its relation to NPI (Coolen and Coolen-Schrijner 2006). However, from Bayesian perspective, there is no strongly convincing argument for such a particular choice of prior distribution, yet the precise choice of prior distribution always influences the inferences to some extent. Even more, the NPI approach is based

on fewer assumptions than the Bayesian approach, as only finite exchangeability is assumed, whereas the assumption of an unknown parameter θ to represent the probability of successful functioning for each component requires an underlying assumption of an infinite population of such components, all of which assumed to be exchangeable (De Finetti 1974). The impact of this latter assumption is often not clear, and might appear to be of little relevance as our NPI results in this paper agree with some Bayesian results, as just mentioned. However, it should be emphasized that the finite exchangeability assumption which underlies the NPI approach does not lead to the assumed existence of a single parameter θ representing the probability of successful functioning of each component considered. One could argue, therefore, that the Bayesian approach uses a de-tour, via a stronger exchangeability assumption on an infinite population of components, to, in this case, get to a similar answer as provided by the more direct NPI approach, with different Bayesian prior distributions corresponding to the NPI lower and upper probabilities. We consider it an advantage of the NPI approach that the inferences are in terms of lower and upper probabilities, as these for example naturally reflect the amount of information available, and deal in an attractive manner with situations where all test results are failures or all are successes. In practical risk assessment, it is often clear which of the lower and upper probabilities should be used to support decisions, while the difference between corresponding upper and lower probabilities can provide further useful information.

The NPI approach for Bernoulli random quantities (Coolen 1998) has been used for several other applications, for example for multiple comparisons of proportions (Coolen and Coolen-Schrijner 2007), where also particular attention has been paid to reliability data with few or zero failures (Coolen-Schrijner and Coolen 2007). The NPI approach for system reliability, as presented in this paper, can be extended to more general system configurations, which provides interesting research challenges. For example, systems consisting of different types of components can be considered, which is a relatively straightforward extension. More challenging is development of the NPI approach for systems consisting of parallel and series subsystems, as for such systems the basic NPI results by Coolen (1998) must be extended to take the particular groupings of future components in the system into account. The basic idea of the NPI approach (Coolen 1998) will remain the same, but the combinatorics involved in deriving the lower and upper probabilities will be challenging for larger systems.

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